ON THE RELATION BETWEEN B.I.B. AND P.B.I.B. DESIGNS

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1. Balanced Incomplete Block (b.i.b.) designs with $v = s^N$, $k = s^m$ and

$$v = \frac{s^{N+1} - 1}{s - 1}, \; k = \frac{s^{m+1} - 1}{s - 1}$$

can always be constructed by taking $(N - m)$-flats as blocks and points as varieties from the spaces $EG(N, p^n = s)$ and $PG(N, p^n = s)$ respectively when

$$\lambda = \frac{(s^{N-1} - 1) \ldots (s^{N-m+1} - 1)}{(s^{m-1} - 1) \ldots (s - 1)}$$

(Bose, 1939). While introducing Partially Balanced Incomplete Block (p.b.i.b.) designs Bose and Nair (1939) have pointed out that in certain cases p.b.i.b. designs can be formed by cutting out one point and all $(N - m)$-flats passing through this point from the space $EG(N, p^n = s)$ or $PG(N, p^n = s)$, and then taking the retained $(N - m)$-flats as our blocks and the retained points as our varieties. Thus from the geometrical b.i.b. designs with $\lambda = 1$; $k = s^2 + s + 1$, or s^2 and $\lambda = s + 1$ p.b.i.b. designs can always be obtained by cutting out a variety and all the blocks containing the variety. It has also been shown by them that p.b.i.b. designs can be formed by cutting out

(*a*) all the points lying on a line, and all planes passing through this line of $PG(3, p^n)$, and then identifying our varieties with the retained points and the blocks with the retained planes (p. 361);

(*b*) all points lying on a line, and all lines passing through points of this line of $PG(3, p^n)$, and then identifying the varieties with points and blocks with lines (p. 362); and

(*c*) all points on three non-concurrent lines, and all lines through the points of intersection of these lines of $PG(2, p^n)$ two by two, and then identifying the blocks with retained straight lines and the varieties with the retained points (p. 363).

Recently Shrikhande (1952 *a*) has mentioned that p.b.i.b. designs can be obtained by cutting out a particular variety and all the blocks containing that variety from the symmetrical b.i.b. designs with $k = 4$, 5, 6, 8, 9, 10 and $\lambda = 1$. It has also been observed by him that p.b.i.b. designs can be obtained by cutting out a variety and all the blocks containing that variety from the b.i.b. designs with

$$v = b = (k^2 - k + 2)/2, \; r = k, \; \lambda = 2$$

The object of the present paper is to study general relation between b.i.b. and p.b.i.b. designs. It will be shown that in certain cases there exist one-to-one correspondence between b.i.b. and p.b.i.b. designs, and one design can easily be constructed from the other.

2. *Lemma I.—Suppose we have a design with* v.r. *units of* v *varieties, each variety being replicated* r *times in* $b = \sum_{L=1}^{2} b_L$ *blocks, each of the* b_L *blocks being of size* k_L $(L = 1, 2)$, *such that every pair of varieties occurs exactly* λ *times. If now* b_1 *blocks constitute a* p.b.i.b. *design with parameters*

$$\left.\begin{array}{l} v' = v, \; b' = b_1, \; r' = r_1, \; k' = k_1, \\ n_i, \; \lambda_i, \; p^i_{jj'}, \; i, j, j' = 1, 2, \ldots, m \end{array}\right\} \qquad (2.1)$$

then the remaining b_2 *blocks of the original design will always be a* p.b.i.b. *design with parameters*

$$\left.\begin{array}{l} v^* = v, \; b^* = b_2, \; r^* = r - r_1, \; k^* = k_2 \\ n_i^* = n_{m-i+1}, \; \lambda_i^* = \lambda - \lambda_{m-i+1}, \\ p^i_{jj'}{}^* = p^{m-i+1}{}_{m-j+1\; m-j'+1}, \; i, j, j' = 1, 2, \ldots, m \end{array}\right\} \qquad (2.2)$$

The proof directly follows from the fact that the two parts of the original design together must be a design in which every pair of varieties should occur λ times and therefore any variety Q which occurs λ_i times with a selected variety ϕ in one part must occur $\lambda - \lambda_i$ times in the other part. Similar argument holds for $p^i_{jj'}$'s.

Here it may be mentioned that in some cases when some λ_i is zero one of the two parts (2.1) and (2.2) may be *disconnected* but both the parts can never be so simultaneously. This is of practical importance as the necessary and sufficient condition for every treatment contrast to be estimable is that the design should be *connected* (Bose, 1947).

Corollary.—Corresponding to every p.b.i.b. *design with parameters* v, b, r, k, n_i, λ_i, $p^i_{jj'}$, i, j, $j' = 1, 2, \ldots, m$ *and in which there is no two identical blocks, there is another* p.b.i.b. *design with parameters.*

$$v^* = v,\ b^* = {}^vC_k - b,\ r^* = {}^{v-1}C_{k-1} - r,\ k^* = k$$

$$n_i^* = n_{m-i+1},\ \lambda_i^* = {}^{v-2}C_{k-2} - \lambda_{m-i+1}$$

$$p^i_{jj'}{}^* = p^{m-i+1}{}_{m-j+1\, m-j'+1},\ i, j, j' = 1, 2, \ldots \ldots, m.$$

In connection with the Lemma I it is interesting to note the following Lemma, the proof of which is obvious.

Lemma II.—Suppose we have a b.i.b. *design with* v, b, r, k, λ *and a* p.b.i.b. *design with* v, b′, r′, k, n_i, λ_i, $p^i_{jj'}$, i, j, j′ = 1, 2, . . ., m, *then by taking the two designs together we can always derive a* p.b.i.b. *design with* $v^* = v$, $b^* = b + b'$, $r^* = r + r'$, k, n_i, $\lambda_i^* = \lambda + \lambda_i$, $p^i_{jj'}{}^* = p^i_{jj'}$, i, j, j′ = 1, 2, . . ., m.

Corollary.—Corresponding to every b.i.b. *design with* v = nk, b = nr, r, k, λ *there is a* p.b.i.b. *design with* $v^* = nk$, $b^* = nr + nt$, $r^* = r + t$, $k^* = k$, $n_1 = k - 1$, $n_2 = v - k$, $\lambda_1 = \lambda + t$, $\lambda_2 = \lambda$,

$$p^1_{jj'} = \begin{Vmatrix} k-2 & 0 \\ 0 & v-k \end{Vmatrix},\quad p^2_{jj'} = \begin{Vmatrix} 0 & k-1 \\ k-1 & v-2k \end{Vmatrix}$$

which can be obtained simply by adding to the b.i.b. *design,* t *times the set of* n *blocks obtained by writing down the* v = nk *varieties in* n *parts with* k *varieties each.*

Lemma II is of less practical importance as the number of replications in the new p.b.i.b. design has unnecessarily been increased.

3. *Theorem* 1.—*If from a* b.i.b. *design with* v, b, r, k, $\lambda = 1$ $(k > 2)$ *all the blocks containing a particular variety are omitted, then the truncated design will always be a* p.b.i.b. *design with parameters*

$$\left.\begin{aligned} & v' = v - 1,\ b' = b - r,\ r' = r - 1,\ k' = k, \\ & n_1 = v - k,\ n_2 = k - 2,\ \lambda_1 = 1,\ \lambda_2 = 0, \\ & p^1_{jj'} = \begin{Vmatrix} v - 2k + 1 & k - 2 \\ k-2 & 0 \end{Vmatrix},\quad p^2_{jj'} = \begin{Vmatrix} v - k & 0 \\ 0 & k-3 \end{Vmatrix} \end{aligned}\right\} \tag{3.1}$$

Conversely if a p.b.i.b. *design with parameters* (3.1) *exists, then a* b.i.b. *design with parameters* $v = v' + 1$, $b = b' + r' + 1$, $r = r' + 1$, $k = k'$, $\lambda = 1$ *can always be constructed.*

Proof.—Consider the r blocks in which a particular variety Q, say occurs. As $\lambda = 1$, any other variety occurs only once in the set of these r blocks. If we now delete the variety Q, then these r blocks will constitute a disconnected p.b.i.b. design with parameters

$$v'' = v - 1,\ b'' = r,\ r'' = 1,\ k'' = k - 1,$$

$$n_1 = k - 2,\ n_2 = v - k,\ \lambda_1 = 1,\ \lambda_2 = 0,$$

$$p^1_{jj'} = \begin{Vmatrix} k-3 & 0 \\ 0 & v-k \end{Vmatrix},\quad p^2_{jj'} = \begin{Vmatrix} 0 & k-2 \\ k-2 & v-2k+1 \end{Vmatrix}$$

whereas the b.i.b. design will turn into a design with $v^* = v - 1$, $b^* = b' + b''$; $r^* = r' + r''$, in which b' blocks are of size k and b'' blocks are of size $k - 1$, but every variety pair occurs once. Therefore by Lemma I the proof of the first part of the theorem follows.

To prove the converse let us suppose that a p.b.i.b. design with parameters (3.1) is known. Now from $r'\ (k' - 1) = n_1\lambda_1 + n_2\lambda_2$ we have $(r - 1)\ (k - 1) = v - k$ or $r\,(k - 1) = v - 1 = v'$. This shows that v' is a multiple of $k - 1$. In fact $(r' + 1)\ (k - 1) = v'$. Now as the design (3.1) is a *GD* design (Bose and Connor, 1952), the v' varieties can be divided into $r' + 1$ groups, each group containing $k - 1$ varieties such that any two varieties of a group are second associates whereas any two varieties belonging to two different groups are first associates. Let us form $r' + 1$ blocks of size $k - 1$ taking $k - 1$ varieties of a group in a block. Adding a new variety Q to each of these $(r' + 1)$ blocks and keeping the other blocks of (3.1) unchanged we get a b.i.b. design with parameters

$$v = v' + 1,\ b = b' + r' + 1,\ r = r' + 1,\ k = k',\ \lambda = 1.$$

When $k = 2$ the design (3.1) reduces to the b.i.b. design $v' = v - 1$, $b' = b - r$, $r' = r - 1$, $k' = 2$, $\lambda = 1$. In fact by omitting all the blocks containing a particular variety, from a b.i.b. design with v, $b = {}^{v}C_k$, $r = {}^{v-1}C_{k-1}$, $\lambda = {}^{v-2}C_{k-2}$ (in which all the combinations of the varieties have completely been written down) we shall always get a b.i.b. design with

$$v^* = v - 1,\ b^* = {}^{v-1}C_k,\ r^* = {}^{v-2}C_{k-1},\ \lambda^* = {}^{v-3}C_{k-2}.$$

Corollary 1.1.—*The* p.b.i.b. *design with parameters*

$$v = b = 35,\ r = k = 6,\ n_1 = 30,\ n_2 = 4,\ \lambda_1 = 1,\ \lambda_2 = 0$$

$$p^1_{jj'} = \begin{Vmatrix} 25 & 4 \\ 4 & 0 \end{Vmatrix},\quad p^2_{jj'} = \begin{Vmatrix} 30 & 0 \\ 0 & 3 \end{Vmatrix}$$

cannot exist as the b.i.b. *design with* v = 36, b = 42, r = 7, k = 6, $\lambda = 1$ *does not.*

Corollary 2.1.—*The* p.b.i.b. *design with parameters*

$$v = 14,\ b = 28,\ r = 6,\ k = 3,\ n_1 = 12,\ n_2 = 1,\ \lambda_1 = 1,\ \lambda_2 = 0$$

$$p^1_{jj'} = \begin{Vmatrix} 10 & 1 \\ 1 & 0 \end{Vmatrix}, \quad p^2_{jj'} = \begin{Vmatrix} 12 & 0 \\ 0 & 0 \end{Vmatrix}$$

has 69 *sets of independent solutions which can be obtained from the* 69 *independent solutions of the* b.i.b. *design* v = 15, b = 35, r = 7, k = 3, $\lambda = 1$ *worked out by Fisher* (1940).

In connection with the above theorem it is worth noting that if from a non-geometrical solution of a b.i.b. design with $v = s^3$, $k = s^2$, $\lambda = s + 1$ or with $v = s^3 + s^2 + s + 1$, $k = s^2 + s + 1$, $\lambda = s + 1$ all the blocks containing a particular variety are omitted we may not get a p.b.i.b. design as can be seen by omitting all the blocks containing the variety ' 1 ' from the solution $[\alpha_1\alpha_1']_1$ of the b.i.b. design $v = b = 15$, $r = k = 7$, $\lambda = 3$, worked out by Nandi (1946 *b*). Similarly it can be shown that property analogous to (*a*) stated in the introduction does not exist in case of non-geometrical solutions.

4. Shrikhande's observation that p.b.i.b. designs can be obtained by omitting a particular variety and all the blocks containing that variety from the b.i.b. design with $v = b = (k^2 - k + 2)/2$, $r = k$, $\lambda = 2$ follows from the following considerations and Lemma 1.

Utilizing the well-known property that any two blocks of a symmetrical b.i.b. design have just λ treatments in common it can be shown that by omitting a particular variety from the r blocks in which it occurs and all the *b-r* blocks in which it does not occur we are always left with a p.b.i.b. design belonging to the series II of Bose (1951) with parameters

$$v^* = k(k-1)/2,\ b^* = k,\ r^* = 2,\ k^* = k - 1,\ n_1 = 2(k-2),$$
$$n_2 = (k-2)(k-3)/2,\ \lambda_1 = 1,\ \lambda_2 = 0,$$

$$p^1_{jj'} = \begin{Vmatrix} k-2 & k-3 \\ k-3 & (k-3)(k-4)/2 \end{Vmatrix}, \quad p^2_{jj'} = \begin{Vmatrix} 4 & 2(k-4) \\ 2(k-4) & (k-4)(k-5)/2 \end{Vmatrix}.$$

Now it is interesting to examine whether p.b.i.b. designs can be obtained by omitting a variety and all the blocks containing that variety from the b.i.b. designs with

$$v^* = (k-1)(k-2)/2,\ b^* = k(k-1)/2,\ r^* = k,\ k^* = k-2,$$
$$\lambda = 2. \qquad (4.1)$$

The structure of (4.1) has been studied by Nandi (1946 *a*) and it has been shown by him that no two blocks can have more than two varieties in common. From this it is seen that if we consider the set of r^* blocks obtained by omitting a particular variety and all the blocks not containing the variety we must have an arrangement with $v' = (k-1)(k-2)/2 - 1$, $b' = k$, $r' = 2$, $k' = k - 3$ in which no pair of varieties occurs more than once. Let us examine whether this can be a p.b.i.b. design with $\lambda_1 = 1$, $\lambda_2 = 0$. Bose (1951) has shown that for a p.b.i.b. design with two replications and $\lambda_1 = 1$, $\lambda_2 = 0$, $k' > r' = 2$, we must have $v' = b'^2/4$ or $b'(b'-1)/2$. From this and Lemma I it follows that, if $k > 6$, by omitting any variety and all the blocks containing that variety from a design (4.1) we can never get a p.b.i.b. design with two associate classes. Even when $k = 6$, in general this method of omission does not give a p.b.i.b. design as can be seen by omitting the variety '1' and all the blocks containing that variety from the solution $[\alpha_3]$ of $v' = 10$, $b' = 15$, $r' = 6$, $k' = 4$, $\lambda = 2$ worked out by Nandi (1946 *a*); but omitting the variety '10' and all the blocks containing '10' from any of the solutions $[\alpha_{11}]$, $[\alpha_3]$, $[\beta_{21}]$ we get a p.b.i.b. design with parameters

$$v = b = 9,\ r = k = 4,$$
$$n_1 = n_2 = 4,\ \lambda_1 = 2,\ \lambda_2 = 1,$$
$$p^1_{jj'} = \begin{Vmatrix} 1 & 2 \\ 2 & 2 \end{Vmatrix},\quad p^2_{jj'} = \begin{Vmatrix} 2 & 2 \\ 2 & 1 \end{Vmatrix}.$$

By omitting a particular variety and all the blocks containing that variety from the b.i.b. design $v = 6$, $b = 10$, $r = 5$, $k = 3$, $\lambda = 2$ we get the p.b.i.b. design $v = b = 5$, $r = k = 3$, $n_1 = n_2 = 2$,

$$\lambda_1 = 2,\ \lambda_2 = 1,\ p^1_{jj'} = \begin{Vmatrix} 0 & 1 \\ 1 & 1 \end{Vmatrix},\quad p^2_{jj'} = \begin{Vmatrix} 1 & 1 \\ 1 & 0 \end{Vmatrix}.$$

Here it is interesting to note that by omitting a particular variety and all the blocks not containing that variety from the b.i.b. design $v = 6$, $b = 10$, $r = 5$, $k = 3$, $\lambda = 2$ we get the p.b.i.b. design $v = b = 5$, $r = k = 2$, $n_1 = n_2 = 2$, $\lambda_1 = 1$, $\lambda_2 = 0$,

$$p^1_{jj'} = \begin{Vmatrix} 0 & 1 \\ 1 & 1 \end{Vmatrix},\quad p^2_{jj'} = \begin{Vmatrix} 1 & 1 \\ 1 & 0 \end{Vmatrix}.$$

which does not belong to any of the series I, II and III to which all the p.b.i.b. designs with two associate classes and $\kappa > r = 2$ belong (Bose, 1951). The solution for the last mentioned design is (1, 2), (1, 3), (2, 4), (3, 5), (4, 5). It can be shown that there is only one more p.b.i.b. design with two associate classes involving two replications, and for which $k = 2$. This belongs to the series I of Bose (1951) and has parameters

$$v = b = 4, \; r = k = 2,$$

$$n_1 = 2, \; n_2 = 1, \; \lambda_1 = 1, \; \lambda_2 = 0$$

$$p^1_{jj'} = \begin{Vmatrix} 0 & 1 \\ 1 & 0 \end{Vmatrix}, \quad p^2_{jj'} = \begin{Vmatrix} 2 & 0 \\ 0 & 0 \end{Vmatrix}.$$

5. An incomplete block design with $v.r.$ units of v varieties there being r units of each variety, arranged into b blocks of size k each is said to be resolvable when the b blocks can be divided into r sets, each set of blocks containing a complete replication of all varieties. Such a design is called affine resolvable when each block of a set has equal number of varieties in common with each of the blocks not in the set.

Lemma III.—Any affine resolvable incomplete block arrangement with

$$v = n^2m, \; b = 2n, \; r = 2, \; k = nm \; (n, m > 1) \qquad (5.1)$$

is a p.b.i.b. *design with parameters*

$$\left.\begin{aligned} & v = n^2m, \; b = 2n, \; r = 2, \; k = nm, \\ & n_1 = m - 1, \; n_2 = 2m(n-1), \; n_3 = m(n-1)^2, \\ & \lambda_1 = 2, \; \lambda_2 = 1, \; \lambda_3 = 0, \\ & p^1_{jj'} = \begin{Vmatrix} m-2 & 0 & 0 \\ 0 & 2m(n-1) & 0 \\ 0 & 0 & m(n-1)^2 \end{Vmatrix}, \; p^2_{jj'} = \begin{Vmatrix} 0 & m-1 & 0 \\ m-1 & m(n-2) & m(n-1) \\ 0 & m(n-1) & m(n-1)(n-2) \end{Vmatrix}, \\ & p^3_{jj'} = \begin{Vmatrix} 0 & 0 & m-1 \\ 0 & 2m & 2m(n-2) \\ m-1 & 2m(n-2) & m(n-2)^2 \end{Vmatrix}. \end{aligned}\right\} \qquad (5.2)$$

The designs (5.2) have been constructed by Nair (1950) in affine resolvable form. The Lemma III states that any affine resolvable arrangement with parameters (5.1) cannot be other than a p.b.i.b. design with parameters (5.2), the formal proof of which follows from the following considerations:—

Any two blocks of an affine resolvable arrangement have exactly m varieties in common unless they belong to the same replication when they have none in common. This shows that nm varieties occurring in a block of any replication can be divided into n groups of m varieties each, such that in the other replication the m varieties of a group occur together in a block but any two varieties belonging to two different groups do not.

Corollary.—Corresponding to every b.i.b. *design with* v = n²m, b = nr, r, k = nm, λ *there is a* p.b.i.b. *design with*

$$v^* = n^2m,\ b^* = nr + 2nt,\ r^* = r + 2t,\ k^* = nm,$$

$$n_1 = m - 1,\ n_2 = 2m\,(n-1),\ n_3 = m\,(n-1)^2,$$

$$\lambda_1 = \lambda + 2t,\ \lambda_2 = \lambda + t,\ \lambda_3 = \lambda.$$

$$p'_{jj'} = \begin{Vmatrix} m-2 & 0 & 0 \\ 0 & 2m\,(n-1) & 0 \\ 0 & 0 & m\,(n-1)^2 \end{Vmatrix},\quad p^2_{jj'} = \begin{Vmatrix} 0 & m-1 & 0 \\ m-1 & m\,(n-2) & m(n-1) \\ 0 & m(n-1) & m(n-1)(n-2) \end{Vmatrix}.$$

$$p^3_{jj'} = \begin{Vmatrix} 0 & 0 & m-1 \\ 0 & 2m & 2m\,(n-2) \\ m-1 & 2m\,(n-2) & m\,(n-2)^2 \end{Vmatrix}$$

which can be obtained by adding to the b.i.b. *design,* t *times the two sets of* n *blocks formed in the following manner.*

Distribute the n^2m varieties of the design in the n^2 cells of a $n \times n$ square so that every cell gets m varieties. Taking the rows and the columns of this square as blocks we get the two sets of n blocks which is nothing but the p.b.i.b. design (5.2).

In connection with Lemma III it is interesting to note that any affine resolvable incomplete block arrangement with $v = n^2m$, $b = 3n$, $r = 3$, $k = nm$ may not be a p.b.i.b. design, as can be seen from the following example:—

(1, 2, 3, 4, 5, 6), (7, 8, 9, 10, 11, 12),

(1, 2, 3, 7. 8, 9), (4, 5, 6, 10, 11, 12),

(1, 2, 4, 8, 10, 12), (3, 5, 6, 7, 9, 11).

This is an affine resolvable arrangement with $v = 12$, $b = 6$, $r = 3$, $k = 6$, but is not a p.b.i.b. design.

Theorem 2.—*If a solution of a* b.i.b. *design with parameters* (M): $v = nk$, $b = nr$, r, k, λ *exists and the solution is such that it contains a set of* n *blocks with a complete replication of all varieties, then by omitting the set of* n *blocks we shall always get a* p.b.i.b. *design with parameters*

$$v' = nk,\ b' = n(r-1),\ r' = r-1,\ k' = k,$$

$$n_1 = (n-1)k,\ n_2 = k-1,\ \lambda_1 = \lambda,\ \lambda_2 = \lambda - 1, \tag{5.3}$$

$$p^1_{jj'} = \left\Vert \begin{matrix} (n-2)k & k-1 \\ k-1 & 0 \end{matrix} \right\Vert, \quad p^2_{jj'} = \left\Vert \begin{matrix} (n-1)k & 0 \\ 0 & k-2 \end{matrix} \right\Vert.$$

Conversely, if a p.b.i.b. *design with parameters* (5.3) *exists, then a solution for the* b.i.b. design (*M*) *can always be constructed.*

Proof.—A set of n blocks containing a complete replication of all the $v = nk$ varieties is obviously a disconnected *p.b.i.b.* design with parameters

$$v'' = nk,\ b'' = n,\ r'' = 1,\ k'' = k$$

$$n_1 = k-1,\ n_2 = v-k,\ \lambda_1 = 1,\ \lambda_2 = 0,$$

$$p^1_{jj'} = \left\Vert \begin{matrix} k-2 & 0 \\ 0 & (n-1)k \end{matrix} \right\Vert, \quad p^2_{jj'} = \left\Vert \begin{matrix} 0 & k-1 \\ k-1 & (n-2)k \end{matrix} \right\Vert.$$

Therefore, by Lemma I, the proof of the first part of the theorem follows.

To prove the converse, let us suppose that the p.b.i.b. design with parameters (5.3) is known. Now, as the design (5.3) is a *GD* design (Bose and Connor, 1952), we can divide the v varieties into n groups of k varieties each in such a way that all the varieties of a group are mutually second associates whereas any two varieties belonging to two different groups are first associates. Obviously, therefore, by taking k varieties of a group in a block, we can form a set of n blocks with a complete replication of all varieties, which when taken together with the blocks of the solution of (5.3) will give a b.i.b. design (*M*).

Corollary 1.—*A* p.b.i.b. *design with parameters*

$$v = nk,\ b = n^2\lambda,\ r = n\lambda,\ k = (n-1)\lambda + 1,$$

$$n_1 = (n-1)k,\ n_2 = k-1,\ \lambda_1 = \lambda,\ \lambda_2 = \lambda - 1,$$

$$p^1_{jj'} = \left\Vert \begin{matrix} (n-2)k & k-1 \\ k-1 & 0 \end{matrix} \right\Vert, \quad p^2_{jj'} = \left\Vert \begin{matrix} (n-1)k & 0 \\ 0 & k-2 \end{matrix} \right\Vert$$

cannot exist unless λ *is of the form* nt + 1, *as it has been shown elsewhere by the author* (Roy, 1952) *that the* b.i.b. *design* (R): $v = nk$, $b = n(n\lambda + 1)$, $r = n\lambda + 1$, $k = (n - 1)\lambda + 1$, λ *cannot have a single set of* n *blocks providing a complete replication of all varieties unless* λ *is of the form* nt + 1.

Corollary 2.—*The* p.b.i.b. *design with parameters*

$$v = 8,\ b = 12,\ r = 6,\ k = 4,$$

$$n_1 = 4,\ n_2 = 3,\ \lambda_1 = 2,\ \lambda_2 = 1,$$

$$p^1_{jj'} = \begin{Vmatrix} 0 & 3 \\ 3 & 0 \end{Vmatrix}, \quad p^2_{jj'} = \begin{Vmatrix} 4 & 0 \\ 0 & 2 \end{Vmatrix}$$

has exactly three independent solutions as Nandi (1946 b) *has shown that the* b.i.b. *design* $v = 8$, $b = 14$, $r = 7$, $k = 4$, $\lambda = 3$ *has just three independent solutions (namely* $[\alpha_1]$, $[\beta_1]$ *and* $[\gamma]$) *with one or more replications of all varieties.*

Theorem 3.—*If a solution of a* b.i.b. *design with parameters*

$$v = n^2m,\ b = nr,\ r,\ k = nm,\ \lambda\ (m \neq 1) \tag{5.4}$$

contains two sets of n *blocks each, each set providing a complete replication of all varieties in such a way that each block of a set has equal number of varieties in common with every block of the other set, then by omitting the two sets of blocks we shall always get a* p.b.i.b. *design with parameters*

$$v' = n^2m,\ b' = n(r-2),\ r' = r-2,\ k' = nm \tag{5.5}$$

$$n_1 = m(n-1)^2,\ n_2 = 2m(n-1),\ n_3 = m-1,$$

$$\lambda_1 = \lambda,\ \lambda_2 = \lambda - 1,\ \lambda_3 = \lambda - 2$$

$$p^1_{jj'} = \begin{Vmatrix} m(n-2)^2 & 2m(n-2) & m-1 \\ 2m(n-2) & 2m & 0 \\ m-1 & 0 & 0 \end{Vmatrix}, \quad p^2_{jj'} = \begin{Vmatrix} m(n-1)(n-2) & m(n-1) & 0 \\ m(n-1) & m(n-2) & m-1 \\ 0 & m-1 & 0 \end{Vmatrix},$$

$$p^3_{jj'} = \begin{Vmatrix} m(n-1)^2 & 0 & 0 \\ 0 & 2m(n-1) & 0 \\ 0 & 0 & m-2 \end{Vmatrix}.$$

The proof of the theorem follows from the Lemmas III and I.

It was expected that the converse of the Theorem 3 will also be true (*Science and Culture*, 1953, **19**, 40–41) but it has not been possible to prove this converse in its complete generality. The present position stands as follows:—

Existence of (5.5) *implies the Existence of* (5.4) *when* $n \neq 4$. *The Case of* $n = 4$ *requires Further Investigation.*

Proof.—(i) $p^3_{33} = m - 2 = n_3 - 1$ shows that every variety with its $m - 1$ third associates form a complete sub-group such that any two varieties of the sub-group are third associates, and as such the n^2m varieties of the design are divisible into n^2 sub-groups each containing m varieties which are mutually third associates.

(ii) $p^3_{22} = n_2$ shows that varieties which are second associates of a chosen variety are also second associates of all the third associates of the chosen variety.

(iii) Similarly $p^3_{11} = n_1$ shows that varieties which are first associates of a chosen variety are also first associates of all the third associates of the chosen variety. (It may be noted that any two of the three conditions imply the third.)

From the above considerations it follows that all the n^2m varieties are always divisible into n^2 sub-groups of m varieties each such that the varieties of a sub-group are mutually third associates whereas all of them are either first or second associates of all the varieties of any other sub-group. This means that if θ is a variety belonging to a sub-group θ_{11} then the remaining $n^2 - 1$ sub-groups of third associates are divisible into two sets—one containing $2(n - 1)$ sub-groups varieties of which are all second associates of the varieties of θ_{11}, and the other containing $(n - 1)^2$ sub-groups varieties of which are all first associates of all the varieties of θ_{11}. This shows that in respect of association we can speak in terms of a sub-group in the sense of the varieties of the sub-group.

Let ϕ be a second associate of θ, and θ_{12} be the sub-group to which ϕ belongs. $p^2_{22} = (n - 2)m$ shows that among the $2n - 3$ sub-groups (excluding θ_{12}) of the second associates of θ_{11} there are just $n - 2$ sub-groups which are also second associates of θ_{12}. Let these $n - 2$ sub-groups be $\theta_{13}, \theta_{14}, \ldots, \theta_{1n}$ and the remaining $(n - 1)$ sub-groups of second associates of θ, *i.e.*, of θ_{11} be $\theta_{21}, \theta_{31}, \ldots, \theta_{n1}$. It may be noted that all of $\theta_{21}, \theta_{31}, \ldots, \theta_{n1}$ are first associates of θ_{12}.

Let ψ be a variety belonging to θ_{i1} $(i > 1)$. Obviously ϕ and ψ are first associates. Now θ_{11} being second associates of both ϕ and ψ, there cannot be more than one sub-group among $\theta_{13}, \theta_{14}, \ldots, \theta_{1n}$ varieties of which are second associates of ψ due to the restriction $p^1{}_{22} = 2m$. Again when there is one sub-group, θ_{1t}, say, whose varieties are second associates of ψ, there must be just one sub-group, θ_{s1}, say, among $\theta_{21}, \theta_{31}, \ldots, \theta_{i-1}, \theta_{i+1}, \ldots, \theta_{n1}$ whose varieties are first associates of ψ, as ψ must have just $(n-1)\,m$ first associates among the second associates of θ [as $p^2{}_{12} = m\,(n-1)$].

On the other hand, for similar reasons, in such a situation θ_{s1} can have no first associate among $\theta_{21}, \theta_{31}, \ldots, \theta_{s-11}, \theta_{s+11}, \ldots, \theta_{n1}$, other than θ_{i1}. In such a case, therefore, θ_{s1} and θ_{i1} have at least θ_{j1} $(j = 2, 3, \ldots, n$, but $j \neq i$ or $s)$ $[(n-2)\,m$ varieties in number] as their common second associates. This is possible only when $p^1{}_{22} \geqslant (n-2)\,m$, *i.e.*, when $n \leqslant 4$, as $p^1{}_{22} = 2m$. But it is easy to see that such a situation cannot arise when $n < 4$. Thus it follows that when $n \neq 4$, θ_{i1} (for all $i > 1$) has all of θ_{1t} $(t = 2, \ldots, n)$ as its first associates. Consequently, and as $p^2{}_{22} = (n-2)\,m$, any two of $(\theta_{11}, \theta_{21}, \theta_{31}, \ldots, \theta_{n1})$ or of $(\theta_{11}, \theta_{12}, \ldots, \theta_{1n})$ are second associates. If $\theta_{i2}, \theta_{i3}, \ldots, \theta_{in}$ be the remaining $(n-1)\,m$ second associates of θ_{i1}, it also follows that all of them are first associates of $\theta_{i'1}$ $(i \neq i')$ and any two of $(\theta_{i1}, \theta_{i2}, \ldots, \theta_{in})$ are second associates. Similarly if $\theta_{2t}, \theta_{3t}, \ldots, \theta_{nt}$ be the remaining $(n-1)\,m$ second associates of θ_{1t}, all of $\theta_{2t}, \theta_{3t}, \ldots, \theta_{nt}$ are first associates of $\theta_{1t'}$ $(t \neq t')$ and any two of $(\theta_{1t}, \theta_{2t}, \ldots, \theta_{nt})$ are second associates. Moreover as θ_{i1} and θ_{1t} must have one sub-group of second associates in common in addition to θ_{11}, without loss of generality θ_{it} can be taken as the common sub-group of second associates of θ_{i1} and θ_{1t}. From what has been discussed above it may be noted that the design (5.5) has properties by virtue of which its n^2m varieties, when $n \neq 4$, can be arranged in the following $n \times n$ schematic arrangement (θ_{it} representing a complete sub-group of third associates as indicated above) such that any two varieties occurring in a cell are mutually third associates, any two varieties occurring in two different cells of a row or of a column are second associates, and any two varieties not occurring in a cell or in cells of a row or of a column are first associates.

Now if we form a set of $2n$ blocks by taking the rows and columns of the scheme as blocks we obviously get an arrangement of the n^2m varieties each being replicated twice, in blocks of size nm such that any two third associates occur together twice, any two second associates occur together once and the first associates do not occur together at all. This shows that addition of this set of $2\,n$ blocks to the design (5.5) will give the design (5.4).

θ_{11}	θ_{12}		θ_{1t}		θ_{1n}
θ_{21}	θ_{22}		θ_{2t}		θ_{2n}
θ_{i1}	θ_{i2}		θ_{it}		θ_{in}
θ_{n1}	θ_{n2}		θ_{nt}		θ_{nn}

When $n = 4$, in addition to the above schematic distribution of varieties showing their association there remains a possibility of the following scheme of association for the varieties of (5.5). Whether any design belonging to (5.5) can actually be constructed with this association scheme has not yet been looked into. If such a design be possible to construct then obviously the converse of the Theorem 3 will not hold good in that case.

Corollary.—The p.b.i.b. *design with parameters*

$$v = 8,\ b = 10,\ r = 5,\ k = 4,$$

$$n_1 = 2,\ n_2 = 4,\ n_3 = 1,\ \lambda_1 = 3,\ \lambda_2 = 2,\ \lambda_3 = 1,$$

$$p^1_{jj'} = \begin{Vmatrix} 0 & 0 & 1 \\ 0 & 4 & 0 \\ 1 & 0 & 0 \end{Vmatrix},\quad p^2_{jj'} = \begin{Vmatrix} 0 & 2 & 0 \\ 2 & 0 & 1 \\ 1 & 0 & 0 \end{Vmatrix}\quad p^3_{jj'} = \begin{Vmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{Vmatrix}$$

has just two independent solutions—one affine resolvable and the other with only one replication of all varieties, as Nandi (1946 b) *has shown that the* b.i.b. *design* v = 8, b = 14, r = 7, k = 4, λ = 3 *has just two*

Second Possible Scheme of Association for (5.5) *when* n = 4.

(For convenience of representation, $\hat{\theta}_{it}$ has been taken as a representative variety of the sub-group θ_{it} and $\bar{\theta}_{it}$ has been taken as the set of the remaining m-1 varieties of θ_{it})

	Variety	Third Associates	Second Associates	First Associates
1	$\hat{\theta}_{11}$	$\bar{\theta}_{11}$	$\theta_{12}, \theta_{13}, \theta_{14}, \theta_{21}, \theta_{31}, \theta_{41}$	$\theta_{22}, \theta_{23}, \theta_{24}, \theta_{32}, \theta_{33}, \theta_{34}, \theta_{42}, \theta_{43}, \theta_{44}$
2	$\hat{\theta}_{12}$	$\bar{\theta}_{12}$	$\theta_{11}, \theta_{13}, \theta_{14}, \theta_{22}, \theta_{32}, \theta_{42}$	$\theta_{21}, \theta_{31}, \theta_{41}, \theta_{23}, \theta_{33}, \theta_{43}, \theta_{24}, \theta_{34}, \theta_{44}$
3	θ_{21}	$\bar{\theta}_{21}$	$\theta_{11}, \theta_{13}, \theta_{31}, \theta_{23}, \theta_{33}, \theta_{43}$	$\theta_{12}, \theta_{14}, \theta_{41}, \theta_{22}, \theta_{32}, \theta_{42}, \theta_{24}, \theta_{34}, \theta_{44}$
4	$\hat{\theta}_{13}$	$\bar{\theta}_{13}$	$\theta_{11}, \theta_{12}, \theta_{21}, \theta_{22}, \theta_{23}, \theta_{24}$	$\theta_{14}, \theta_{31}, \theta_{41}, \theta_{32}, \theta_{42}, \theta_{33}, \theta_{43}, \theta_{34}, \theta_{44}$
5	$\hat{\theta}_{14}$	$\bar{\theta}_{14}$	$\theta_{11}, \theta_{12}, \theta_{41}, \theta_{32}, \theta_{33}, \theta_{34}$	$\theta_{21}, \theta_{13}, \theta_{31}, \theta_{22}, \theta_{42}, \theta_{23}, \theta_{43}, \theta_{24}, \theta_{44}$
6	$\hat{\theta}_{31}$	$\bar{\theta}_{31}$	$\theta_{11}, \theta_{21}, \theta_{41}, \theta_{42}, \theta_{43}, \theta_{44}$	$\theta_{12}, \theta_{13}, \theta_{14}, \theta_{22}, \theta_{32}, \theta_{23}, \theta_{33}, \theta_{24}, \theta_{34}$
7	$\hat{\theta}_{41}$	$\bar{\theta}_{41}$	$\theta_{11}, \theta_{14}, \theta_{31}, \theta_{24}, \theta_{34}, \theta_{44}$	$\theta_{12}, \theta_{21}, \theta_{13}, \theta_{22}, \theta_{32}, \theta_{42}, \theta_{23}, \theta_{33}, \theta_{43}$
8	$\hat{\theta}_{22}$	$\bar{\theta}_{22}$	$\theta_{12}, \theta_{13}, \theta_{42}, \theta_{43}, \theta_{24}, \theta_{34}$	$\theta_{11}, \theta_{21}, \theta_{14}, \theta_{31}, \theta_{41}, \theta_{32}, \theta_{23}, \theta_{33}, \theta_{44}$
9	$\hat{\theta}_{23}$	$\bar{\theta}_{23}$	$\theta_{21}, \theta_{13}, \theta_{24}, \theta_{32}, \theta_{33}, \theta_{44}$	$\theta_{11}, \theta_{12}, \theta_{14}, \theta_{31}, \theta_{41}, \theta_{22}, \theta_{34}, \theta_{42}, \theta_{43}$
10	$\hat{\theta}_{24}$	$\bar{\theta}_{24}$	$\theta_{13}, \theta_{41}, \theta_{22}, \theta_{23}, \theta_{34}, \theta_{44}$	$\theta_{11}, \theta_{12}, \theta_{21}, \theta_{14}, \theta_{31}, \theta_{32}, \theta_{33}, \theta_{42}, \theta_{43}$
11	$\hat{\theta}_{32}$	$\bar{\theta}_{32}$	$\theta_{12}, \theta_{14}, \theta_{23}, \theta_{33}, \theta_{42}, \theta_{44}$	$\theta_{11}, \theta_{13}, \theta_{21}, \theta_{31}, \theta_{41}, \theta_{22}, \theta_{24}, \theta_{34}, \theta_{43}$
12	$\hat{\theta}_{33}$	$\bar{\theta}_{33}$	$\theta_{14}, \theta_{21}, \theta_{23}, \theta_{32}, \theta_{34}, \theta_{43}$	$\theta_{11}, \theta_{12}, \theta_{13}, \theta_{31}, \theta_{41}, \theta_{22}, \theta_{24}, \theta_{42}, \theta_{44}$
13	$\hat{\theta}_{34}$	$\bar{\theta}_{34}$	$\theta_{14}, \theta_{41}, \theta_{22}, \theta_{24}, \theta_{33}, \theta_{43}$	$\theta_{11}, \theta_{12}, \theta_{13}, \theta_{21}, \theta_{31}, \theta_{23}, \theta_{42}, \theta_{32}, \theta_{44}$
14	$\hat{\theta}_{42}$	$\bar{\theta}_{42}$	$\theta_{12}, \theta_{31}, \theta_{22}, \theta_{32}, \theta_{43}, \theta_{44}$	$\theta_{11}, \theta_{13}, \theta_{14}, \theta_{21}, \theta_{41}, \theta_{23}, \theta_{24}, \theta_{33}, \theta_{34}$
15	$\hat{\theta}_{43}$	$\bar{\theta}_{43}$	$\theta_{21}, \theta_{31}, \theta_{22}, \theta_{42}, \theta_{33}, \theta_{34}$	$\theta_{11}, \theta_{12}, \theta_{13}, \theta_{14}, \theta_{41}, \theta_{23}, \theta_{24}, \theta_{32}, \theta_{44}$
16	$\hat{\theta}_{44}$	$\bar{\theta}_{44}$	$\theta_{31}, \theta_{41}, \theta_{23}, \theta_{24}, \theta_{32}, \theta_{42}$	$\theta_{11}, \theta_{12}, \theta_{13}, \theta_{14}, \theta_{21}, \theta_{22}, \theta_{33}, \theta_{34}, \theta_{43}$

independent solutions (namely $[\gamma]$ *and* $[\beta_1]$) with $L \geqslant 2$ *replications of all varieties.*

6. Shrikhande (1952 *b*) has pointed out that by omitting a complete replication from the affine resolvable design $v = s^2$, $b = s^2 + s$, $r = s + 1$, $k = s$, $\lambda = 1$, and the treatments lying in any n $(< s - 1)$ blocks of the omitted replication, we get a group divisible design with $v = s(s - n)$, $b = s^2$, $r = s$, $k = s - n$, where the v varieties can be divided into $s - n$ groups of s each, where any two treatments of the same group do not occur together in any block, whereas any two treatments coming from different groups occur together in just one block. Now I shall prove a theorem which is more general than the result obtained by Shrikhande.

Theorem 4.—*If we have a solution of a* b.i.b. *design with parameters* (D): $v = n^2(n-1)t + n^2$, $b = n(n^2t + n + 1)$, $r = n^2t + n + 1$, $k = n(n-1)t + n$, $\lambda = nt + 1$, *such that there is a set of* n *blocks providing a complete replication of all varieties, then by suppressing the set of* n *blocks and the varieties occurring in any* $n - L$ $(1 < L \leqslant n)$ *blocks of the set, we shall always get a* p.b.i.b. *design with parameters*

$$v' = L\{n(n-1)t + n\},\ b' = n^2(nt+1),\ r' = n(nt+1),$$

$$k' = L\{(n-1)t + 1\},$$

$$n_1 = n\{(n-1)t+1\}(L-1),\ n_2 = (n-1)(nt+1),$$

$$\lambda_1 = nt + 1,\ \lambda_2 = nt,$$

$$p^1_{jj'} = \begin{Vmatrix} n\{(n-1)t+1\}(L-2) & (n-1)(nt+1) \\ (n-1)(nt+1) & 0 \end{Vmatrix},$$

$$p^2_{jj'} = \begin{Vmatrix} n\{(n-1)t+1\}(L-1) & 0 \\ 0 & (n-1)(nt+1)-1 \end{Vmatrix}.$$

Proof.—It has been shown elsewhere by the author (Roy, 1952) that if a b.i.b. design belonging to the series (*D*) has a set of n blocks providing a complete replication of all varieties, then each block of the set has an equal number of varieties [namely $(n-1)t + 1$] in common with each of the blocks not in the replication. From this fact and Lemma I, the proof of the above theorem follows.

We shall conclude by proving a theorem regarding an affine resolvable arrangement from which some interesting properties about the b.i.b. design $v = 8$, $b = 14$, $r = 7$, $k = 4$, $\lambda = 3$ will follow.

Theorem 5.—Any affine resolvable arrangement with v = 8, b = 6, r = 3, k = 4 is a p.b.i.b. *design with parameters*

$$n_1 = 1,\ n_2 = 6,\ \lambda_1 = 3,\ \lambda_2 = 1,$$

$$p^1_{jj'} = \begin{Vmatrix} 0 & 0 \\ 0 & 6 \end{Vmatrix}, \quad p^2_{jj'} = \begin{Vmatrix} 0 & 1 \\ 1 & 4 \end{Vmatrix} \tag{6.1}$$

or

$$n_1 = 3,\ n_2 = 3,\ n_3 = 1,\ \lambda_1 = 2,\ \lambda_2 = 1,\ \lambda_3 = 0,$$

$$p^1_{jj'} = \begin{Vmatrix} 0 & 2 & 0 \\ 2 & 0 & 1 \\ 0 & 1 & 0 \end{Vmatrix}, \quad p^2_{jj'} = \begin{Vmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{Vmatrix}, \quad p^3_{jj'} = \begin{Vmatrix} 0 & 3 & 0 \\ 3 & 0 & 0 \\ 0 & 0 & 0 \end{Vmatrix} \tag{6.2}$$

Proof.—Let us consider about any variety ϕ_1. Let B_{i1} ($i = 1, 2, 3$) be the block belonging to the i-th replication in which ϕ_1 occurs and B_{i2} be the other block of the i-th replication. Suppose the contents of the blocks B_{11} and B_{21} are $(\phi_1\phi_2\phi_3\phi_4)$ and $(\phi_5\phi_6\phi_7\phi_8)$ respectively. As every block of a replication has exactly two varieties in common with each of the blocks not in the replication, any two of the blocks B_{11}, B_{21} and B_{31} have exactly one variety other than ϕ_1 in common. In this connection, only two situations can arise: If ϕ_2 be the second variety (ϕ_1 being the first) common between B_{11} and B_{21}, then B_{31} may or may not contain the variety ϕ_2.

(i) Suppose B_{11}, B_{21} and B_{31} have the two varieties ϕ_1 and ϕ_2 in common. In this case it should be noted that, as every block of a replication must have exactly two varieties in common, the contents of the six blocks with respect to the occurrence and non-occurrence of the other varieties so far as $(\phi_1\phi_2)$ and $(\phi_3\phi_4)$ are concerned are uniquely fixed up as follows:—

$$\begin{aligned} &B_{11}\,(\phi_1\phi_2\phi_3\phi_4);\ B_{12}\,(\phi_5\phi_6\phi_7\phi_8) \\ &B_{21}\,(\phi_1\phi_2\phi_5\phi_6);\ B_{22}\,(\phi_3\phi_4\phi_7\phi_8) \\ &B_{31}\,(\phi_1\phi_2\phi_7\phi_8);\ B_{32}\,(\phi_3\phi_4\phi_5\phi_6) \end{aligned} \tag{6.3}$$

(Interchange of the varieties ϕ_5, ϕ_6, ϕ_7, ϕ_8 is immaterial so far as the varieties ϕ_1, ϕ_2, ϕ_3 and ϕ_4 are concerned.)

The affine resolvable arrangement (6.3) is obviously the p.b.i.b. design with parameters (6.2).

(ii) Suppose B_{11} and B_{21} have the two varieties ϕ_1 and ϕ_2 in common but B_{31} does not contain ϕ_2. As B_{31} must have exactly two treatments in common with all the blocks except B_{32}, without loss of generality the contents of the six blocks can be taken as

$$B_{11}\ (\phi_1\phi_2\phi_3\phi_4);\ B_{12}\ (\phi_5\phi_6\phi_7\phi_8)$$

$$B_{21}\ (\phi_1\phi_2\phi_5\phi_6);\ B_{22}(\phi_3\phi_4\phi_7\phi_8)$$

$$B_{31}\ (\phi_1\phi_3\phi_5\phi_7);\ B_{32}\ (\phi_2\phi_4\phi_6\phi_8)$$

Except interchange of treatments, this affine resolvable arrangement is uniquely fixed up and is a p.b.i.b. design with parameters (6.1).

Corollary 1.—Any R $(2 \leqslant R \leqslant 6)$ replications of any solution of the b.i.b. design $v = 8$, $b = 14$, $r = 7$, $k = 4$, $\lambda = 3$ is an affine resolvable p.b.i.b. design.

Corollary 2.—By omitting $R\,(1 \leqslant R \leqslant 5)$ replications from any solution of the b.i.b. design $v = 8$, $b = 14$, $r = 7$, $k = 4$, $\lambda = 3$, we shall always be left with a p.b.i.b. design (which may or may not be resolvable).

SUMMARY

The present investigation deals with combinatorial relation between certain balanced and partially balanced incomplete block designs, the whole argument being based on the result that *addition* (*or subtraction*) *of a partially balanced incomplete block design to* (*or from*) *a balanced incomplete block design with the same number of varieties and the same block size leads to a partially balanced incomplete block design.* This is in continuation of the two earlier communications of the author—one in the *Bulletin of the Calcutta Mathematical Society* and the other in '*Sankhyā*', the *Indian Journal of Statistics*, in which *duality relation* between these two types of designs have fully been discussed. The object of all these studies were to examine how the two types of designs—namely balanced and partially balanced incomplete block designs are related.

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